

ON NEEMAN'S GRADIENT FLOWS

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To Jim Lepowsky and Robert Wilson with admiration.

ABSTRACT. In his brilliant but sketchy paper on the structure of quotient varieties of affine actions of reductive algebraic groups over \mathbb{C} Amnon Neeman introduced a gradient flow with remarkable properties. The purpose of this paper is to study several applications of this flow. In particular we prove that the cone on a Zariski closed subset of $\mathbb{P}^{n-1}(\mathbb{R})$ is a deformation retract of \mathbb{R}^n . We also give an exposition of an extension to real reductive algebraic group actions of Schwarz's excellent explanation of Neeman's sketch of a proof of his deformation theorem. This exposition precisely explains the use of Lojasiewicz gradient inequality. The result described above for cones makes use of these ideas.

1. INTRODUCTION

The purpose of this note is to give an exposition of how an idea of Amnon Neeman [N] (and Mumford) and results of Lojasiewicz [L] can be used to prove some topological results for real projective varieties. For example, it is proved that the affine cone on a Zariski closed subspace of real projective space is a deformation retract of \mathbb{R}^n (see Theorem 11 in section 2). These ideas were applied to geometric invariant theory over \mathbb{C} by Neeman implying that if G is a reductive group over \mathbb{C} acting on \mathbb{C}^n and K is a maximal compact subgroup of G (which we can assume acts unitarily) and if X is a G -invariant subvariety of \mathbb{C}^n then the Kempf-Ness set [KN] of X is a strong K -equivariant deformation retract of X . We give an argument for the corresponding result over \mathbb{R} (see also Richardson-Solovay [RS]). There is a complete exposition of this aspect of the work in the paper of Schwarz [S] (emphasizing the theory over \mathbb{C}). Anyone who has attempted to read Neeman's paper ([N]), owes a debt of gratitude to the careful exposition in [S]. [N] contains a weak form of the deformation theorem in its first two sections. In sections four and later which contain the more sophisticated topology Neeman mainly uses the weak form. Section three contains the ideas mentioned above. In that section a sketch of the proof of the deformation theorem is given on the basis of a "conjecture of Mumford"

(3.1 in the paper) which he extends by making another conjecture (3.5). In the introduction Neeman writes:

“Now let us say something about Section 3. When I wrote the paper it was a largely conjectural section, but now I know that both Conjecture 3.1 and Conjecture 3.5 are true. Conjecture 3.5 is a special case of an inequality due to Łojasiewicz, and Conjecture 3.1 can be proved from Łojasiewicz’s inequality using estimates similar to those in Section 3. I chose not to rewrite the text, because at present I do not feel I could give an adequate account of the proof of Conjecture 3.1. Although Łojasiewicz’s inequality is enough, a stronger inequality should be true; roughly speaking, I conjecture that the correct value for ε in Conjecture 3.5 is $1/2$ (see remark 3.7). For this reason I feel the appendix is still important; it contains evidence for my new conjectures. If I rewrote Section 3 to incorporate my new conjectures, the new section would be too long, and largely unconnected with the rest of the paper.”

In this paper we expand a bit on the exposition of [S] and prove a stronger form of “Conjecture 3.1” (following Neeman’s suggestion). Neeman also conjectured that the correct ε is $\frac{1}{2}$. Neeman gives a sketch of an argument in the case of tori (alluded to in the quote) which we expand in the last section. We observe that his argument doesn’t use the Łojasiewicz theory to get the stronger result.

The result of Łojasiewicz involves mathematics outside of the usual universe of researchers in the theory of algebraic groups involving the study of real algebraic (and analytic) inequalities initiated in the Tarski-Seidenberg theorem (c.f. [H]) and expanded on in Łojasiewicz in his development of real analytic geometry ([L]). Since this theory is also far away from my expertise, I show, in the last section, that some of the ideas that only involve freshman calculus can be used to prove useful weaker results.

2. SOME GRADIENT SYSTEMS

Let $\phi \in \mathbb{R}[x_1, \dots, x_n]$ be a polynomial that is homogeneous of degree m such that $\phi(x) \geq 0$ for all $x \in \mathbb{R}^n$. We consider the gradient system

$$\frac{dx}{dt} = -\nabla\phi(x)$$

relative to the usual inner product on \mathbb{R}^n , $\langle x, y \rangle = \sum x_i y_i$. Where, as usual,

$$\nabla\phi(x) = \sum \frac{\partial\phi}{\partial x_i} e_i$$

with $\{e_1, \dots, e_n\}$ the standard orthonormal basis. Then

$$\langle \nabla \phi(x), x \rangle = m\phi(x).(*)$$

So, if we denote by $F(t, x)$ the solution to the system for t near $t = 0$ with $F(t, 0) = x$ then

$$\frac{d}{dt} \langle F(t, x), F(t, x) \rangle = -2 \langle \nabla \phi(F(t, x)), F(t, x) \rangle = -2m\phi(F(t, x)) \leq 0.$$

This implies

Lemma 1. $\|F(t, x)\| \leq \|x\|$ if $F(s, x)$ is defined for $0 \leq s \leq t$.

We therefore have

Lemma 2. $F(t, x)$ is defined for all $t \geq 0$, $x \in \mathbb{R}^n$ and smooth in (t, x) .

Proof. Assume that $F(t, x)$ is defined for $0 \leq t < t_o$. Let $\{t_j\}$ be a sequence in $[0, t_o)$ with $\lim_{j \rightarrow \infty} t_j = t_o$. Then since $\|F(t_j, x)\| \leq \|x\|$ there is an infinite subsequence $\{t_{j_k}\}$ such that $\{F(t_{j_k}, x)\}$ converges to x_o . Let $\varepsilon > 0$ be such that $F(s, y)$ is defined and smooth on $|s| < \varepsilon$ and $(-\varepsilon, \varepsilon) \times B_\varepsilon(x_o)$ ($B_r(y)$ is the usual Euclidean r -ball with center y). There exists N such that if $k \geq N$ then $|t_{j_k} - t_o| < \varepsilon$ and $\|F(t_{j_k}, x) - x_o\| < \varepsilon$. Fix $k \geq N$. Then $t_{j_k} = t_o - s$ with $|s| < \varepsilon$ and $\|F(t_o - s, x) - x_o\| < \varepsilon$. Thus if $\delta = |\varepsilon - |s||$ and $|u| < \varepsilon$ then $F(s + u, F(t_o - s, x))$ is defined. Hence $F(t_o + u, x)$ is defined for $|u| < \delta$ and given by $F(s + u, F(t_o - s, x))$. \square

The formula $(*)$ combined with the Schwarz inequality implies

Lemma 3. $\|\nabla \phi(x)\| \|x\| \geq m\phi(x)$. Thus if $\|x\| \leq r$ then

$$\|\nabla \phi(x)\| \geq \frac{m}{r} \phi(x).$$

The Lojasiewicz gradient inequality [L] implies the following improvement of the equality in the above Lemma.

Theorem 4. Assume that $m > 1$. There exists $0 < \varepsilon \leq \frac{1}{m-1}$ and $C > 0$ both depending only on ϕ such that for all $x \in \mathbb{R}^n$

$$\|\nabla \phi(x)\|^{1+\varepsilon} \|x\|^{1-(m-1)\varepsilon} \geq C\phi(x).$$

To see this we recall the Lojasiewicz inequality

Theorem 5. If ψ is a real analytic function on an open subset, U , of \mathbb{R}^n and if $x_o \in U$ then there exist $C > 0$, $\varepsilon > 0$ and $r > 0$ such that $B_r(x_o) = \{x \in \mathbb{R}^n \mid \|x - x_o\| < r\} \subset U$ and

$$\|\nabla \psi(x)\|^{1+\varepsilon} \geq C|\psi(x) - \psi(x_o)|$$

if $x \in B_r(x_o)$.

To prove the asserted implication we note since $\phi(0) = 0$ there exist ε and r as in the theorem above so that

$$\|\nabla\phi(x)\|^{1+\varepsilon} \geq C|\phi(x)|, x \in B_r(0).$$

If $\varepsilon > \frac{1}{m-1}$ we argue that we may replace ε with any $0 < \delta \leq \frac{1}{m-1}$. Since $\nabla\phi(0) = 0$ we can choose $s \leq r$ such that if $\|x\| < s$ then $\|\nabla\phi(x)\| \leq 1$ hence if $\|x\| < s$, $\|\nabla\phi(x)\|^{1+\delta} \geq \|\nabla\phi(x)\|^{1+\varepsilon}$. Thus we may assume $0 < \varepsilon \leq \frac{1}{m-1}$. We now may scale in x (using the fact that $\nabla\phi$ is homogeneous of degree $m-1$) to see that with a different constant C we have

$$\|\nabla\phi(x)\|^{1+\varepsilon} \geq C|\phi(x)|, x \in \overline{B_1(0)}.$$

Thus if $\|x\| = 1$ we have

$$\|\nabla\phi(x)\|^{1+\varepsilon} \|x\|^{1-(m-1)\varepsilon} \geq C|\phi(x)|.$$

Noting that the homogeneity of the left hand side is

$$1 + \varepsilon(m-1) + 1 - (m-1)\varepsilon = m$$

the theorem now follows. Since ϕ is homogeneous of degree m . One is tempted, on the basis of homogeneity, to think that $\varepsilon = \frac{1}{m-1}$ would be the correct choice in the theorem above. This is related to Neeman's remark 3.7 as mentioned in the introduction.

3. THE NEEMAN FLOW (AS EXPLAINED BY GERRY SCHWARZ)

We use the notation of the previous section. We take ε and C as above (but note that one can very simply get the estimate in the theorem with $\varepsilon = 0$). If we write F for $F(t, X)$ and $H(t) = \phi(F(t, x))$ then we have

$$H'(t) = -d\phi(F)(\nabla\phi(F)) = -\|\nabla\phi(F)\|^2.$$

If $t \geq 0$ and $\|x\| \leq r$

$$\|\nabla\phi(F)\|^{1+\varepsilon} \|F\|^{1-(m-1)\varepsilon} \geq C\phi(F).$$

Thus

$$\|\nabla\phi(F)\|^{1+\varepsilon} \geq \frac{C}{r^{1-(m-1)\varepsilon}} \phi(F).$$

Hence

$$\|\nabla\phi(F)\|^2 \geq \left(\frac{C}{r^{1-(m-1)\varepsilon}} \right)^{\frac{2}{1+\varepsilon}} \phi(F)^{\frac{2}{1+\varepsilon}}.$$

Thus

$$|H'(t)| \geq \frac{1}{2} \left(\frac{C}{r^{1-(m-1)\varepsilon}} \right)^{\frac{2}{1+\varepsilon}} \phi(F)^{\frac{2}{1+\varepsilon}} = C_1(r) H(t)^{\frac{2}{1+\varepsilon}}.$$

This yields (since $H'(t) \leq 0$)

$$-H'(t) \geq C_1(r)H(t)^{\frac{2}{1+\varepsilon}}.$$

Thus

$$\frac{d}{dt}H(t)^{-\frac{1}{1+\varepsilon}} = -\frac{H'(t)}{H(t)^{\frac{2}{1+\varepsilon}}} \geq C_1(r)$$

we conclude that if $t > 0$ then

$$H(t)^{-\frac{1}{1+\varepsilon}} \geq C_1(r)t.$$

Inverting we have

$$H(t) \leq C_2(r)t^{-(1+\varepsilon)}$$

with $C_2(r) = C_1(r)^{-(1+\varepsilon)}$. The result of Lojasiewicz gains us the $\varepsilon > 0$. The key aspect of this inequality is that the only dependence is on r so it is true for any $F(t, x)$ with $\|x\| \leq r$ and $t > 0$. In many cases the easy case $\varepsilon = 0$ is sufficient. We now show how the $\varepsilon > 0$ leads to an important result (the argument is modeled on the exposition of G. Schwarz [S]).

We note that the above inequality implies that if $f(t) = t^{1+\delta}$ with $0 < \delta < \varepsilon$ then for $t > 0$

$$0 < H(t)f'(t) \leq C_2(r)(1+\delta)t^{-1-(\varepsilon-\delta)}.$$

Let $0 < t < s$ then

$$\begin{aligned} H(s)f(s) - H(t)f(t) &= \int_t^s \frac{d}{du}(H(u)f(u))du = \\ &= \int_t^s H(u)f'(u)du + \int_t^s H'(u)f(u)du. \end{aligned}$$

Thus

$$-\int_t^s H'(u)f(u)du = \int_t^s H(u)f'(u)du + H(t)f(t) - H(s)f(s).$$

We also note that

$$0 \leq H(s)f(s) \leq C_2(r)s^{-(1+\varepsilon)}s^{1+\delta} = C_2(r)s^{-(\varepsilon-\delta)}.$$

Since $|H'(u)| = -H'(u)$ this implies

$$\lim_{s \rightarrow +\infty} \int_t^s |H'(u)|f(u)du = \int_t^\infty H(u)f'(u)du + H(t)f(t) < \infty.$$

Thus $\sqrt{|H'(u)|f(u)}$ is in $L^2([t, +\infty))$ for all $t > 0$ and so

$$|H'(u)| = \sqrt{|H'(u)|f(u)}u^{-\frac{(1+\delta)}{2}} \in L^1([t, +\infty)).$$

All estimates are uniform for $\|x\| \leq r < \infty$ so we have proved:

Theorem 6. *If $t > 0$ then*

$$\int_t^{+\infty} \left\| \frac{d}{du} F(u, x) \right\| du$$

converges uniformly for $\|x\| \leq r$.

This result implies that if $t \geq 0$ then

$$\int_t^\infty \frac{d}{du} F(u, x) du$$

converges absolutely and uniformly for $\|x\| \leq r < \infty$. Noting that if $s > t$ then

$$\int_t^s \frac{d}{du} F(u, x) du = F(s, x) - F(t, x)$$

we have for $t > 0$

$$\lim_{s \rightarrow \infty} F(s, x) = \int_t^\infty \frac{d}{du} F(u, x) du + F(t, x).$$

So if we set $U(t, x) = F(\frac{t}{1-t}, x)$ and define $U(1, x)$ by the limit above then $U : [0, 1] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is continuous and

$$\nabla \phi(x) = 0 \iff \phi(x) = 0$$

(Lemma 3 and the fact that 0 is a minimum for ϕ) we have proved

Theorem 7. *$U : [0, 1] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ defines a strong deformation retraction of \mathbb{R}^n onto $Y = \{x \in \mathbb{R}^n | \phi(x) = 0\}$.*

Proof. We note since $\nabla \phi(y) = 0$ if $y \in Y$ then $F(t, y) = y$ for all $y \in Y$. Thus $U(0, x) = x$ all $x \in \mathbb{R}^n$, $U(t, y) = y$ all $0 \leq t \leq 1$ and all $y \in Y$ and since

$$\lim_{t \rightarrow +\infty} \phi(F(t, x)) = 0$$

we have $U(1, \mathbb{R}^n) = Y$. □

A deformation retraction of a topological space X onto a closed subspace Y is a continuous map $U : [0, 1] \times X \rightarrow X$ such that $U(1, X) = Y$ and $U(t, y) = y$ for all $y \in Y$ and $t \in [0, 1]$.

We now derive a few corollaries to this result. The first is obvious.

Corollary 8. *If $X \subset \mathbb{R}^n$ is a closed subset such that $F(t, X) \subset X$ for all $t \geq 0$ then $Y \cap X$ is a strong deformation retraction of X .*

Corollary 9. *Let K be a compact subgroup of $GL(n, \mathbb{R})$ and assume that $\phi(kx) = \phi(x)$ for $k \in K, x \in \mathbb{R}^n$. If X is as above and invariant under K then the strong retraction in the previous corollary is K equivariant.*

Proof. We note that the K -invariance of ϕ implies that $\nabla\phi(kx) = k\nabla\phi(x)$ for $k \in K, x \in \mathbb{R}^n$. Thus

$$\frac{d}{dt}k^{-1}F(t, kx) = -k^{-1}\nabla\phi(F(t, kx)) - \nabla\phi(k^{-1}F(t, kx))$$

and since

$$k^{-1}F(0, kx) = x$$

the uniqueness theorem implies that

$$k^{-1}F(t, kx) = F(t, x).$$

□

We now assume that $Y \subset \mathbb{R}^n$ is the locus of zeros of homogeneous polynomials f_1, \dots, f_m with $\deg f_i = r_i$. We set $r = \text{lcm}(r_1, \dots, r_m)$ and

$$\phi(x) = \sum_{i=1}^m (f_i^{\frac{r}{r_i}})^2.$$

Then $Y = \{x \in \mathbb{R}^n | \phi(x) = 0\}$. Let $F(t, x)$ be as above for this choice of ϕ . Then we can apply the Corollaries to this case.

Finally, let K be a compact subgroup of $GL(n, \mathbb{R})$ and $KY \subset Y$ with Y the zero locus of f_i for f_i as above.

Lemma 10. *Define $\phi_K(x) = \int_K \phi(kx) dk$ then ϕ_K is a homogeneous polynomial of degree $2r$, $\phi_K(x) \geq 0$ all $x \in \mathbb{R}^n$ and $Y = \{x \in \mathbb{R}^n | \phi_K(x) = 0\}$.*

Proof. We note that

$$\int_K \phi(kx) dk = \sum_{i=1}^m \int_K (f_i(kx))^{\frac{2r}{r_i}} dk.$$

Thus since each integrand is non-negative if $\phi_K(x) = 0$ then we have for all i

$$\int_K (f_i(kx))^{\frac{2r}{r_i}} dk = 0$$

and hence $f_i(kx) = 0$ for all k and i . Hence $x \in Y$. The lemma is now obvious. □

Combining this with the above Corollary we have

Theorem 11. *If $X \subset \mathbb{P}^{n-1}(\mathbb{R})$ is a K invariant Zariski closed then there exists a K -equivariant strong deformation retract of \mathbb{R}^n to the cone on X in \mathbb{R}^n .*

4. NEEMAN'S THEOREM.

We now look at the main example for which the conditions of the above corollaries are satisfied.

Let G be a real algebraic subgroup of $GL(n, \mathbb{R})$ invariant under transpose and let $K = G \cap O(n)$. Let for $x \in \mathbb{R}^n$, $X \in \mathfrak{g} = Lie(G)$

$$f_x(X) = \langle Xx, x \rangle$$

then $f_x \in \mathfrak{g}^*$. On \mathfrak{g}^* we put the inner product dual to $(X, Y) = \text{tr}(XY^*)$ (here Y^* is just the transpose of Y). Then we take

$$\phi(x) = \|f_x\|^2.$$

Looking upon \mathbb{R}^n as $n \times 1$ matrices we have

$$f_x(X) = \text{tr}(Xxx^*).$$

Hence $f_x(X)$ is the inner product of X with $P_{\mathfrak{g}}(xx^*)$ where $P_{\mathfrak{g}}$ is the orthogonal projection of $M_n(\mathbb{R})$ onto \mathfrak{g} . So

$$\phi(x) = \text{tr}(P_{\mathfrak{g}}(xx^*)^2).$$

We now compute the gradient of ϕ

$$\begin{aligned} d\phi_x(v) &= 2\text{tr}P_{\mathfrak{g}}(vx^* + xv^*)P_{\mathfrak{g}}xx^* = \\ 2\text{tr}((vx^* + xv^*)P_{\mathfrak{g}}xx^*) &= 2\langle v, P_{\mathfrak{g}}(xx^*)x \rangle + 2\langle x, P_{\mathfrak{g}}(xx^*)v \rangle = \\ &= 4\langle v, P_{\mathfrak{g}}(xx^*)x \rangle \end{aligned}$$

since $P_{\mathfrak{g}}(xx^*)^* = P_{\mathfrak{g}}(xx^*)$. Thus

$$\nabla\phi(x) = 4P_{\mathfrak{g}}(xx^*)x \in T_x(Gx).$$

This implies that $F(t, x) \in Gx$ for all $t \geq 0$.

To put this in context we recall the Kempf-Ness theorem over \mathbb{R} . Then $v \in \mathbb{R}^n$ will be said to be *critical* if $\langle Xv, v \rangle = 0$ for all $X \in \mathfrak{g} = Lie(G)$. We note that this is the same as saying that $\langle Xv, v \rangle = 0$ for all $X \in \mathfrak{p} = \{Y \in \mathfrak{g} | Y^* = -Y\}$. Here is the Kempf-Ness theorem in this context (the topological assertions are for the subspace topology in \mathbb{R}^n).

Theorem 12. *Let G, K be as above. Let $v \in \mathbb{R}^n$.*

1. *v is critical if and only if $\|gv\| \geq \|v\|$ for all $g \in G$.*
2. *If v is critical and $X \in \mathfrak{p}$ is such that $\|e^X v\| = \|v\|$ then $Xv = 0$. If $w \in Gv$ is such that $\|v\| = \|w\|$ then $w \in Kv$.*
3. *If Gv is closed then there exists a critical element in Gv .*
4. *If v is critical then Gv is closed.*

We set $Crit_G(\mathbb{R}^n)$ equal to the real algebraic variety of critical elements. We note that $Crit_G(\mathbb{R}^n)$ is the zero set of $\phi(x) = \text{tr}P_{\mathfrak{g}}(xx^*)^2$.

We can now state the theorem of Neeman over \mathbb{R} .

Theorem 13. *Let X be a G -invariant closed subset of \mathbb{R}^n then $X \cap \text{Crit}(\mathbb{R}^n)$ is a strong K -equivariant deformation retract of X .*

Proof. We note that $\phi(x) = \text{tr} P_g(xx^*)^2$ is K -invariant and $F(t, x) \in Gx$ thus any G -invariant subset of \mathbb{R}^n is invariant under the flow. The theorem follows from Corollary 9. \square

In the course of our proof of this version of the Kempf-Ness theorem we proved an auxiliary result (see [W], 3.6.2). Let $G_{\mathbb{C}}$ be the Zariski closure of G in $GL(n, \mathbb{C})$ then $G_{\mathbb{C}}$ is invariant under adjoint and hence is reductive. Let $L = G_{\mathbb{C}} \cap U(n)$ then L is a maximal compact subgroup of $G_{\mathbb{C}}$ and $L \cap G = K$. The Kempf-Ness theorem (in the complex case) implies that if $v \in \mathbb{C}^n$ is $G_{\mathbb{C}}$ -critical then $G_{\mathbb{C}}v \cap \text{Crit}(\mathbb{C}^n) = Uv$. The following result was proved

Proposition 14. *If $v \in \mathbb{R}^n$ is G -critical then it is $G_{\mathbb{C}}$ critical and $G_{\mathbb{C}}v \cap \mathbb{R}^n$ is a finite union of open G -orbits (hence closed).*

We note that this shows that 4.in the Kempf-Ness theorem over \mathbb{C} implies 4.in the theorem over \mathbb{R} (the rest is just calculus).

Corollary 15. *If $v \in \mathbb{R}^n$ is G -critical then $Lv \cap \mathbb{R}^n = Kv_1 \cup \dots \cup Kv_r$ a finite number of K -orbits.*

Proof. Since $G_{\mathbb{C}}v \cap \mathbb{R}^n$ is closed, the above proposition and 3. in the Kempf-Ness theorem imply that $G_{\mathbb{C}}v \cap \mathbb{R}^n = \cup_{j=1}^r Gv_j$ with v_j critical in \mathbb{R}^n . Since $G_{\mathbb{C}}v \cap \text{Crit}(\mathbb{C}^n) = Lv$, and $\text{Crit}(\mathbb{C}^n) \cap \mathbb{R}^n = \text{Crit}(\mathbb{R}^n)$ we have

$$\begin{aligned} Lv \cap \mathbb{R}^n &= \left(\cup_{j=1}^r Gv_j \right) \cap \text{Crit}(\mathbb{R}^n) = \\ \cup_{j=1}^r (Gv_j \cap \text{Crit}(\mathbb{R}^n)) &= Kv_1 \cup \dots \cup Kv_r. \end{aligned}$$

\square

The r in the statement can be larger than 1. This is the reason why the next section is over \mathbb{C} .

5. AN ELEMENTARY RESULT

We retain the notation of the previous section. In this section we explain how the elementary estimate (that only uses Freshman calculus)

$$\phi(F(t, x)) \leq \frac{C(\|x\|)}{t}$$

for $t > 0$ can prove a useful weakening of Neeman's theorem for actions of connected reductive algebraic groups over \mathbb{C} . Let $G \subset GL(n, \mathbb{C})$ be Zariski closed and invariant under adjoint. Let K be the intersection of G with $U(n)$. We look upon \mathbb{C}^n as $\mathbb{R}^{2n} = \mathbb{R}^n \oplus i\mathbb{R}^n$ and G as a real

algebraic group. Thus K is also the intersection of G with $O(2n)$. In this context if $v \in \mathbb{R}^{2n}$ then \overline{Gv} contains a unique closed orbit and $\overline{Gv} \cap \text{Crit}(\mathbb{R}^{2n})$ is a single K -orbit. We also note that $F(t, kv) = kF(t, v)$. Thus F induces a flow on \mathbb{R}^{2n}/K , which we denote by $H(t, Kx)$.

We note

Theorem 16. *Let $v \in \mathbb{C}^n$ then $\lim_{t \rightarrow +\infty} H(t, Kv) = \overline{Gv} \cap \text{Crit}(\mathbb{R}^{2n}) = Ku$.*

Proof. The above estimate implies that

$$\lim_{t \rightarrow +\infty} \phi(F(t, v)) = 0.$$

We have also seen that if $t > 0$, then $\|F(t, v)\| \leq \|v\|$. Let $\{t_j\}$ be a sequence in $\mathbb{R}_{>0}$ such that $\lim_{j \rightarrow \infty} t_j = +\infty$. The sequence $\{F(t_j, v)\}$ is bounded. Let $F(t_{j_k}, v)$ be a convergent subsequence. Then $\lim_{k \rightarrow \infty} F(t_{j_k}, v) = u \in \overline{Gv}$ and $\phi(u) = 0$. Thus $Ku = \overline{Gv} \cap \text{Crit}(\mathbb{R}^{2n})$. Thus every convergent subsequence of $\{H(t_j, Kv)\}$ converges to Ku . This implies the theorem. \square

6. NEEMAN'S ARGUMENT FOR TORI

As indicated in the introduction Neeman conjectured that in the context of Section 4 (there ϕ is homogeneous of degree 4) there should exist $C > 0$ such that for all x

$$C \|\nabla \phi(x)\|^{\frac{4}{3}} \geq \phi(x).$$

As evidence for this assertion he gave a sketch of a proof for the case when G (in that section is commutative). We will devote this section to filling out his brilliant proof this case. We first set up the general question. Let G be a closed subgroup of $GL(N, \mathbb{R})$ such that G is invariant under adjoint. Let $\mathfrak{p} = \{X \in \text{Lie}(G) | X^* = X\}$. We have seen that if P is the orthogonal projection of $M_N(\mathbb{R})$ onto \mathfrak{p} (here we are using the inner product $\text{tr}XY^*$) then $\phi(x) = \text{tr}(P(xx^*))^2$ (here we look upon x as an $N \times 1$ column). Now if X_1, \dots, X_n is an orthonormal basis of \mathfrak{p} then

$$P(xx^*) = \sum_i \text{tr}(X_i xx^*) X_i = \sum_i \langle X_i x, x \rangle X_i$$

and

$$\phi(x) = \sum_i \langle X_i x, x \rangle^2.$$

We also note that

$$\nabla \phi(x) = 4 \sum_i \langle X_i x, x \rangle X_i x.$$

Hence

$$\|\nabla\phi(x)\|^2 = \sum_{i,j} \langle X_i v, v \rangle \langle X_j v, v \rangle \langle X_i v, X_j v \rangle.$$

Thus the theorem below implies the desired result for the case when G is abelian. The following lemma plays an important role in the proof of the theorem and since it may not be well known so we include a proof before embarking on the proof of the theorem.

Let $(V, \langle \dots, \dots \rangle)$ be a finite dimensional inner product space over \mathbb{R} .

Lemma 17. *Let $v_1, \dots, v_n \in V$ spanning an m -dimensional vector space. Then there exists $A = [a_{ij}]_{1 \leq i, j \leq n}$ an orthogonal matrix over \mathbb{R} and c_1, \dots, c_k in $\mathbb{R}_{>0}$ such that if $z_i = \sum_j a_{ij} v_j$ then $z_j = 0$ for $j > m$ and*

$$\langle z_i, z_j \rangle = \delta_{ij} c_i, 1 \leq i, j \leq m.$$

Proof. After permuting the v_j we may assume that v_1, \dots, v_k are linearly independent. Let

$$v_{m+j} = \sum_{i=1}^m x_{j,i} v_i.$$

Let X be the $n - m$ by m matrix with entries x_{ij} . We form the block matrix

$$B = \begin{bmatrix} I_m & 0 \\ X & -I_{n-m} \end{bmatrix} = [b_{ij}]$$

with I_r the $r \times r$ identity matrix. Then $\sum_j b_{ij} v_j = v_i$ for $i \leq m$ and $\sum_j b_{ij} v_j = 0$ for $i > m$. Using the Iwasawa decomposition for $GL(n, \mathbb{R})$ (i.e. Gram-Schmidt) we can write

$$B = uak$$

with u upper triangular with 1's on the main diagonal, a diagonal with positive diagonal entries a_1, \dots, a_n and $k \in O(n)$. We have

$$B \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} \sum b_{1j} v_j \\ \vdots \\ \sum b_{mj} v_j \end{bmatrix} = \begin{bmatrix} v_1 \\ \vdots \\ v_m \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

So

$$ak \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = u^{-1} B \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = u^{-1} \begin{bmatrix} v_1 \\ \vdots \\ v_m \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} w_1 \\ \vdots \\ w_m \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

with w_1, \dots, w_m linearly independent. Now apply a^{-1} and have

$$k \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} a_1^{-1} w_1 \\ \vdots \\ a_m^{-1} w_m \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} t_1 \\ \vdots \\ t_m \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Finally, we choose an orthogonal $m \times m$ matrix T that diagonalizes the form

$$\sum_{1 \leq i, j \leq m} x_i \langle t_i, t_j \rangle x_j.$$

Setting

$$S = \begin{bmatrix} T & 0 \\ 0 & I \end{bmatrix}$$

then $A = Sk$ is the desired orthogonal transformation. \square

Corollary 18. *Let $X_1, \dots, X_n \in \text{End}(V)$ and $v \in V$. Suppose that the span of $\{X_i v\}$ has dimension m . Then there exists $A = [a_{ij}] \in O(n)$ such that if $Z_i = \sum a_{ij} X_j$ then $Z_i v = 0$ for $i > m$ and $\langle Z_i v, Z_j v \rangle = \delta_{ij} c_i$ with $c_i > 0$ for $i \leq m$.*

Proof. Apply the above lemma to $v_i = X_i v$, $i = 1, \dots, n$. \square

We note that if X_1, \dots, X_n are self adjoint elements of $\text{End}(V)$ and $\phi(x) = \sum_{i=1}^n \langle X_i v, v \rangle^2$ then $\nabla \phi(x) = 4 \sum_{i=1}^n \langle X_i v, v \rangle X_i v$. In this case the homogeneity is $m = 4$ and thus the suggested strong form of the inequality is

$$C \|\nabla \phi(x)\|^{1+\frac{1}{3}} \geq \phi(x).$$

The following theorem of Neeman proves this result if the X_i mutually commute. We include a detailed proof following Neeman's sketch since this result is so suggestive. We also make clear where the commutivity

assumption is used (exactly one step). In the proof we will use the obvious identity

$$\left\| \sum_{i=1}^n \langle X_i v, v \rangle X_i v \right\|^2 = \sum_{i,j} \langle X_i v, v \rangle \langle X_j v, v \rangle \langle X_i v, X_j v \rangle$$

Theorem 19. *Let $\{X_1, \dots, X_n\}$ be a set of self adjoint elements of $\text{End}(V)$ such that $[X_i, X_j] = 0$ for $1 \leq i, j \leq n$. There exists a constant $C > 0$ such that of $v \in V$ then*

$$C \left(\sum_{i,j} \langle X_i v, v \rangle \langle X_j v, v \rangle \langle X_i v, X_j v \rangle \right)^2 \geq \left(\sum_i \langle X_i v, v \rangle^2 \right)^3.$$

Proof. Let S be the unit sphere in V . We note that the Theorem follows from the following local version.

(*) If $v_o \in S$ then there exists a neighborhood Ω_v of v in and C_v such that

$$C_v \left(\sum_{i,j=1}^n \langle X_i x, x \rangle \langle X_j x, x \rangle \langle X_i x, X_j x \rangle \right)^2 \geq \left(\sum_{i=1}^n \langle X_i x, x \rangle^2 \right)^3, x \in \Omega_v.$$

Indeed, since S is compact we can choose a finite number $v_1, \dots, v_r \in S$ such that $\cup \Omega_{v_i}$ cover S . Choose $C = \max_{1 \leq i \leq r} C_{v_i}$.

We will now prove (*) by induction on n . If $n = 1$ then we write X for X_1 and we may assume that X is diagonal. If $X = 0$ then the theorem is obvious. So assume $X \neq 0$ then we may take an orthonormal basis v_1, \dots, v_N of V such that $X v_i = a_i v_i$ with $a_i \in \mathbb{R}$, $a_i \neq 0$ for $i = 1, \dots, k$ and $a_i = 0$ for $i > k$ and $|a_i| \geq |a_{i+1}|$. Now if $v = \sum x_i v_i$ then

$$\langle X v, v \rangle \langle X v, v \rangle \langle X v, X v \rangle = \langle X v, v \rangle^2 \sum a_i^2 x_i^2 \geq$$

$$a_k^2 \langle X v, v \rangle^2 \sum_{i=1}^k x_i^2 \geq \frac{a_k^2}{|a_1|} \langle X v, v \rangle^2 \sum |a_i| x_i^2 \geq \frac{a_k^2}{|a_1|} \langle X v, v \rangle^2 |\langle X v, v \rangle|.$$

This proves the theorem for $n = 1$ hence (*) in this case.

Now we assume that (*) is true for $1 \leq k < n$ and we prove it for n . If $\cap \ker X_i \neq (0)$ then the theorem follows from the case when V is replaced by $Z = (\cap \ker X_i)^\perp$ and the X_i are replaced by $X_i|_Z$. Thus we may assume that $\cap \ker X_i = (0)$. We are now ready to prove the inductive step. Consider $v_o \in S$.

Let $B(v)$ denote the $n \times n$ matrix with i, j entry $\langle X_i v, X_j v \rangle$. Suppose that $v_o \in V$ is such that $X_1 v_o, \dots, X_n v_o$ are linearly independent. Then $B(v_o)$ is positive definite. Thus there is a compact neighborhood, U ,

of v_0 in S and $C_1 > 0$ such that $B(v_0) - C_1 I$ is positive semidefinite. Thus on U we have

$$\sum_{i,j} \langle X_i v, v \rangle \langle X_j v, v \rangle \langle X_i v, X_j v \rangle \geq C_1 \sum_i \langle X_i v, v \rangle^2.$$

We note that there is a positive constant C_2 such that if $v \in S$ then $|\langle X_i v, v \rangle| \leq C_2 \langle v, v \rangle = C_2$. So

$$\left(\sum_i \langle X_i v, v \rangle^2 \right)^{\frac{1}{2}} \leq \sqrt{n} C_2.$$

Thus on U we have

$$\sum_{i,j} \langle X_i v, v \rangle \langle X_j v, v \rangle \langle X_i v, X_j v \rangle \geq \frac{C_1}{\sqrt{n} C_2} \left(\sum_i \langle X_i v, v \rangle^2 \right)^{\frac{3}{2}}.$$

The desired inequality. We may thus assume that the span of $\{X_i v_o\}_{i=1}^n$ has dimension $1 \leq l < n$.

Let $A = [a_{ij}] \in O(n)$ be as in the corollary above for v_o . We note that $\sum_{i,j} \langle X_i v, v \rangle \langle X_j v, v \rangle \langle X_i v, X_j v \rangle$ and $\sum_i \langle X_i v, v \rangle^2$ are unchanged under the transformation $X_i \rightarrow \sum a_{ij} X_j$. Replacing X_j with $\sum_i a_{ji} X_i$ we may assume that if $l = \dim \text{Span}\{X_1 v_o, \dots, X_n v_o\}$ then $X_i v_o = 0$ for $i > l$ and the $X_i v_o$ for $i \leq l$ are mutually orthogonal. We come now to the only place we use the assumption that $[X_i, X_j] = 0$ for $1 \leq i, j \leq n$.

Let \mathcal{A} denote the algebra generated by the X_i . Let $V_0 = \mathcal{A} v_o$ and let $P : V \rightarrow V_0$ be the orthogonal projection. Then we note that $X_i P = P X_i$ all i and $X_i P = 0$ if $i > l$. Now

$$\begin{aligned} \left\| \sum_{i=1}^n \langle X_i v, v \rangle X_i v \right\| &\geq \left\| \sum_{i=1}^n \langle X_i v, v \rangle P X_i v \right\| = \\ \left\| \sum_{i=1}^n \langle X_i v, v \rangle X_i P v \right\| &= \left\| \sum_{i=1}^l \langle X_i v, v \rangle X_i P v \right\| \end{aligned}$$

Noting that

$$[\langle X_i v_o, X_j v_o \rangle]_{1 \leq i, j \leq l} = [\langle X_i P v_o, X_j P v_o \rangle]_{1 \leq i, j \leq l}$$

is positive definite we see that there exists U be a compact neighborhood of v_o such that

$$B_1(v) = [\langle X_i P v, X_j P v \rangle]_{1 \leq i, j \leq l}$$

is positive definite for $v \in U$. We also note that we can choose a perhaps smaller neighborhood such that

$$B_2(v) = [\langle X_i v, X_j v \rangle]_{1 \leq i, j \leq l}$$

is also positive definite for $\nu \in U$. Thus there is a constant $C_3 > 0$ such that $B_1(v) - C_3 B_2(\nu)$ is positive semidefinite for $v \in U$. So \square

$$\sum_{i,j=1}^n \langle X_i v, v \rangle \langle X_j v, v \rangle \langle X_i v, X_j v \rangle \geq \sum_{i,j=1}^l \langle X_i v, v \rangle \langle X_j v, v \rangle \langle X_i P v, X_j P v \rangle \geq C_3 \sum_{i,j=1}^l \langle X_i v, v \rangle \langle X_j v, v \rangle$$

Set $C_4 = \frac{1}{C_5}$. We have shown that if $v \in U$ then

$$C_4 \left\| \sum_{i=1}^n \langle X_i v, v \rangle X_i v \right\| \geq \left\| \sum_{i=1}^l \langle X_i v, v \rangle X_i v \right\|$$

There are obviously two possibilities for every $v \in S$

- I. $2 \left\| \sum_{i=1}^l \langle X_i v, v \rangle X_i v \right\| \geq \left\| \sum_{i=l+1}^n \langle X_i v, v \rangle X_i v \right\|$ or
- II. $2 \left\| \sum_{i=1}^l \langle X_i v, v \rangle X_i v \right\| < \left\| \sum_{i=l+1}^n \langle X_i v, v \rangle X_i v \right\|$.

We write $a = \left\| \sum_{i=1}^l \langle X_i v, v \rangle X_i v \right\|$, $b = \left\| \sum_{i=l+1}^n \langle X_i v, v \rangle X_i v \right\|$.

We assume that $v \in U$. In case I. Observing that if $a, b \geq 0$ and $2a \geq b$ then $3a = a + 2a \geq a + b$ thus in case I,

$$3C_4 \left\| \sum_{i=1}^n \langle X_i v, v \rangle X_i v \right\| \geq \left(\left\| \sum_{i=1}^l \langle X_i v, v \rangle X_i v \right\| + \left\| \sum_{i=l+1}^n \langle X_i v, v \rangle X_i v \right\| \right)$$

and in case II. We have

$$\left\| \sum_{i=1}^n \langle X_i v, v \rangle X_i v \right\| \geq \left\| \sum_{i=l+1}^n \langle X_i v, v \rangle X_i v \right\| - \left\| \sum_{i=1}^l \langle X_i v, v \rangle X_i v \right\|.$$

This time $b \geq 2a$ then

$$\begin{aligned} b - a &\geq \frac{1}{3}b + \left(\frac{2}{3}b - a \right) = \\ &\frac{1}{3}b + \frac{1}{6}b \geq \frac{1}{3}(a + b). \end{aligned}$$

Thus in case II. We have

$$\left\| \sum_{i=1}^n \langle X_i v, v \rangle X_i v \right\| \geq \frac{1}{3} \left(\left\| \sum_{i=1}^l \langle X_i v, v \rangle X_i v \right\| + \left\| \sum_{i=l+1}^n \langle X_i v, v \rangle X_i v \right\| \right).$$

Thus if C_5 is the maximum of 3 and $3C_4$ we have for all $v \in U$

$$C_5 \left\| \sum_{i=1}^n \langle X_i v, v \rangle X_i v \right\| \geq \left\| \sum_{i=1}^l \langle X_i v, v \rangle X_i v \right\| + \left\| \sum_{i=l+1}^n \langle X_i v, v \rangle X_i v \right\|$$

Since $0 < l < n$ the inductive hypothesis implies that there is an open neighborhood W of v_o in U and a constant $C_6 > 0$ such that

$$\left\| \sum_{i=1}^l \langle X_i v, v \rangle X_i v \right\| + \left\| \sum_{i=l+1}^n \langle X_i v, v \rangle X_i v \right\| \geq C_6 \left(\left(\sum_{i=1}^l \langle X_i v, v \rangle^2 \right)^{\frac{3}{4}} + \left(\sum_{i=l+1}^n \langle X_i v, v \rangle^2 \right)^{\frac{3}{4}} \right).$$

Thus for $v \in W$ we have

$$\begin{aligned} C_5 \left(\sum_{i,j} \langle X_i v, v \rangle \langle X_j v, v \rangle \langle X_i v, X_j v \rangle \right)^2 = \\ C_5 \left\| \sum_{i=1}^n \langle X_i v, v \rangle X_i v \right\|^4 \geq C_6^4 C_5 \left(\left(\sum_{i=1}^l \langle X_i v, v \rangle^2 \right)^{\frac{3}{4}} + \left(\sum_{i=l+1}^n \langle X_i v, v \rangle^2 \right)^{\frac{3}{4}} \right)^4 \geq \\ C_6^4 C_5 \left(\left(\sum_{i=1}^l \langle X_i v, v \rangle^2 \right)^3 + \left(\sum_{i=l+1}^n \langle X_i v, v \rangle^2 \right)^3 \right). \end{aligned}$$

We note that if $a, b \geq 0$ then $a^3 + b^3 \geq \frac{1}{8}(a+b)^3$. We may assume $a \leq b$. Then if $a = 0$ the inequality is obvious so assume that $0 < a \leq b$. Set $x = \frac{b}{a} \geq 1$ then $8 + 8x^3 > 1 + 3x^3 + 3x^3 + x^3 \geq 1 + 3x + 3x^2 + x^2 = (1+x)^3$. Thus

$$C_5 \left\| \sum_{i=1}^n \langle X_i v, v \rangle X_i v \right\|^4 \geq \frac{C_5 C_6^4}{8} \left(\sum_{i=1}^l \langle X_i v, v \rangle^2 + \sum_{i=l+1}^n \langle X_i v, v \rangle^2 \right)^3$$

for $v \in W$. This completes the induction.

Bibliography

- [H] Lars Hörmander, The Analysis of Linear Partial Differential Operators II, Differential Operators with Constant Coefficients, Springer-Verlag, Berlin, 1983, Appendix A, 362–371.
- [KN] George Kempf and Linda Ness, The length of vectors in representation spaces, Algebraic geometry (Proc. Summer Meeting, Univ. Copenhagen, 1978), Lecture Notes in Math. 732, Berlin, New York: Springer-Verlag, 1979 233–243
- [L] S. Lojasiewicz, Ensembles semi-analytiques, Preprint IHES, 1965.
- [N] Amnon Neeman, The topology of quotient varieties, Annals of Math. (122), 1985, 419–459.

- [RS] R. W. Richardson and P. J. Slodowy, Minimum Vectors for Real Reductive Algebraic Groups, *J. London Math. Soc.* (2) 92 (1990), 409–429.
- [S] Gerald W. Schwarz, Topological methods in algebraic transformation groups, 135–151, *Progress in Mathematics*, Volume 80, Birkhäuser, Boston, 1989.
- [W] Nolan R. Wallach, Geometric invariant theory over the real and complex numbers, to appear, Springer.